

Dynamics of quantum entanglement in the reservoir with memory effects

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Abstract

The non-Markovian dynamics of quantum entanglement is studied by the Shabani-Lidar master equation when one of entangled quantum systems is coupled to a local reservoir with memory effects. The completely positive reduced dynamical map can be constructed in the Kraus representation. Quantum entanglement decays more slowly in the non-Markovian environment. The decoherence time for quantum entanglement can be markedly increased by the change of the memory kernel. It is found out that the entanglement sudden death between quantum systems and entanglement sudden birth between the system and reservoir occur at different instants.

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1. Introduction

The quantum entanglement in composite systems is a key issue in quantum information theory [1, 2]. The resources of quantum entanglement are contained in nonclassical states which are useful for quantum information processing, like quantum teleportation, cryptography, dense coding and quantum computation [3–5]. It is known that entanglement can be efficiently quantified by some useful measurement including the concurrence [6], the negativity [7] and relative entropy [8, 9]. Because of interactions between open systems and surrounding environments, the decoherence of quantum entanglement is usually inevitable [10]. Therefore, dynamical properties of quantum entanglement have received much attention [11–24]. In some works [25, 26], the dynamics of the two-qubit entanglement have been investigated when the noninteracting qubits are respectively coupled to two independent reservoirs. It was found out that the sudden death and revival of the entanglement can happen at some instants. The Markovian approximation is used for the study of dynamics of open systems weakly coupled to reservoirs without memory effects. The corresponding map is always completely positive and trace-preserving. However, in realistic environments with memory effects, time evolutions of quantum systems often obey the non-Markovian dynamics [10]. The memory effects can be described by some phenomenological master equations with a reasonable memory kernel [27–31]. It is an obstacle to deal with physical decoherence of open systems analytically and numerically in non-Markovian environments. Among these master equations, the Shabani-Lidar one interpolates between the exact dynamics and Markovian one [29]. The desirable property of Shabani-Lidar master equation is the preservation of the complete positivity of the reduced dynamics. The requirement of complete positivity of dynamics ensures that the physical states of reduced systems can be applied to practical tasks of quantum information processing. From the point of view of a definite physical process, it is of great value to obtain a completely positive dynamical map in a non-Markovian bath and investigate the dynamics of quantum entanglement.

In this paper, by the Shabani-Lidar master equation, we construct the completely-positive dynamical map in order to analyze the decoherence of a two-level quantum system coupled to a vacuum reservoir with general exponential memory. In Sec. 2, the reduced dynamical map is written in the Kraus representation. The dynamics of quantum entanglement between the open qubit and an ancillary system isolated from the local reservoir is obtained in Sec. 3. The memory effects on distribution of quantum entanglement between the systems and

local bath are also taken into account. A simple discussion concludes the paper.

2. The completely positive reduced dynamical map

The decoherence of quantum system is always unavoidable because of the interactions with its local reservoir. To set up the reduced dynamical map, we investigate the total Hamiltonian of the effective two-level open system and the local environment,

$$H = H_S + H_E + H_{SE}, \quad (1)$$

where

$$H_S = \omega_0 \sigma_+ \sigma_- \quad (2)$$

describes the intrinsic Hamiltonian of the open two-level system,

$$H_E = \sum_j \omega_j a_j^\dagger a_j \quad (3)$$

is the Hamiltonian of the reservoir with a large amount of independent harmonic oscillators and

$$H_{SE} = \sum_j g_j (\sigma_+ a_j + \sigma_- a_j^\dagger) \quad (4)$$

represents the interaction between the open qubit S and the environment E . The parameter ω_0 describes the transition energy from the ground state $|g\rangle$ to the excited one $|e\rangle$. The operators $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are the raising and lowering operators of the open qubit. Here a_j^\dagger, a_j are the creation and annihilation operators of the reservoir. In the environment with memory effects, the time evolution of an effective two-level system can be approximately obtained by the Shabani-Lidar master equation [29]. The master equation can give new insight into the dynamics of non-Markovian quantum systems. The reduced states for the open system $\rho_S(t)$ can be written as

$$\frac{\partial \rho_S(t)}{\partial t} = \mathcal{L} \int_0^t f(t') \exp(\mathcal{L}t') \rho_S(t - t') dt'. \quad (5)$$

Here the rotating-wave approximation is adopted. In the following analysis, the initial total state $\rho_S(0) \otimes |0\rangle_E \langle 0|$ is assumed. $|0\rangle_E$ denotes the vacuum state of the local reservoir. The function $f(t')$ is the phenomenological kernel characterizing the memory effects from the environment. The memory kernel function can be experimentally determined by quantum

state tomography [29]. In order to understand the physical meaning of the Shabani-Lidar memory kernel, we stress that this master equation describes a situation in which a measurement of the environment at time t' is followed by a Markovian evolution, described in terms of continuous measurements of the environment at times $t > t'$ [29, 31]. The time t' characterizes the bath memory effects. The memory kernel is a function which assigns weights to different measurements. The determination of the memory kernel is related to the reservoir spectral density. The local Liouvillian operator \mathcal{L} can be expressed by

$$\mathcal{L}\rho_S = \gamma_0[\sigma_- \rho_S \sigma_+ - \frac{1}{2}(\sigma_+ \sigma_- \rho_S + \rho_S \sigma_+ \sigma_-)]. \quad (6)$$

The parameter $\gamma_0 = \gamma_0(g_j, \omega_j, \omega_0)$ describes the decaying rate in the Markovian model. For the dynamical map $\Phi(t)$ of the reduced system S , the density matrix at any time t can also be expressed by $\rho_S(t) = \Phi(t)\rho_S(0)$. According to [29], the definite expression of the dynamical map is given by $\Phi(t)\rho_S(0) = \sum_{j=0}^3 \xi_j(t) \text{Tr}[L_j \rho_S(0)] R_j$. Here L_j and R_j are the j -th left and right damping basis of the Liouvillian operator [32]. In this case, $\xi_j(t) = \text{Lap}^{-1}[\frac{1}{s - \lambda_j F(s - \lambda_j)}]$ where $\{\lambda_j = 1, -\gamma_0, -\gamma_0/2, -\gamma_0/2\}$ are the four eigenvalues of the Liouvillian operator and the Laplace transformation $F(s) = \text{Lap}[f(t)]$.

To investigate the conditions of complete positivity of the dynamics, the Choi matrix \mathcal{P} need be constructed by the elements $\mathcal{P}(i, j) = \Phi(t)|i\rangle\langle j|$ [33]. In the context, the general exponential memory kernel function is considered as $f(t) = A \exp(-\gamma t)$ where A is the amplitude of the kernel and γ represents the memory decaying rate. This kind of memory kernel can be physically modeled by [30]. The definite expression of the matrix \mathcal{P} can be obtained by

$$\mathcal{P} = \begin{pmatrix} \xi(\gamma_0, t) & 0 & 0 & \xi(\gamma_0/2, t) \\ 0 & 1 - \xi(\gamma_0, t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi(\gamma_0/2, t) & 0 & 0 & 1 \end{pmatrix}, \quad (7)$$

where the elements $\xi(x, t)$ can be written as $\xi(x, t) = \exp[-\frac{(1+g)xt}{2}](\cosh x\tilde{\omega}t + \frac{1+g}{2\tilde{\omega}} \sinh x\tilde{\omega}t)$. The parameter is $\tilde{\omega} = |\sqrt{(1+g)^2 - 4\alpha}|/2$ where $\alpha = A/x$ and $g = \gamma/x$. If $\alpha > (1+g)^2/4$, the above function $\xi(x, t)$ is obtained by substituting $\sinh[\cdot]$ and $\cosh[\cdot]$ with $\sin[\cdot]$ and $\cos[\cdot]$.

According to the results of [29], the complete positivity of the map is equivalent to the positivity of the matrix \mathcal{P} where all of the eigenvalues are non-negative. The condition of

complete positivity of the map can be analytically given by the inequality of

$$\xi^2(\gamma_0/2, t) \leq \xi(\gamma_0, t) \leq 1. \quad (8)$$

The impacts of the memory kernel on the complete positivity of the dynamics are numerically calculated in Fig. 1. When $A \leq \gamma$ or $\alpha \leq g$, the values of $D = \xi(\gamma_0, t) - \xi^2(\gamma_0/2, t)$ are always non-negative in Fig. 1(a). Because the values of $\xi(x, t)$ are monotonically decreasing in time, the inequality of $\xi(x, \infty) = 0 \leq \xi(x, t) \leq \xi(x, 0) = 1$ is always satisfied. This means that the corresponding map at any time is completely positive in the conditions of $\alpha \leq g$. However, if the amplitude of the memory kernel satisfies $\alpha > (1 + g)^2/4$, the difference D oscillates between the positive values and negative ones in Fig. 1(b). For the time interval $\gamma_0 t \in (\pi - \arctan \frac{2\tilde{\omega}}{1+g}, 2\pi - \arctan \frac{2\tilde{\omega}}{1+g})$, the values of $\xi(\gamma_0, t)$ are negative which violates the positivity of the map. The region of $D < 0$ represents the non-Markovian dynamical map which contradicts the complete positivity. According to the results of [31], during the time evolution which violates the positivity condition, the density matrix loses the probabilistic interpretation. The reduced states in this regions are not physical. Therefore, the small amplitude of the kernel function is useful for setting up the completely-positive dynamical map. In the following analysis, we consider the physical evolution of the reduced states in the region of $\alpha \leq g$.

In the condition of the complete positivity, the reduced dynamical map can be expressed in the Kraus representation,

$$\rho_S(t) = \sum_k \hat{M}_k \rho_S(0) \hat{M}_k^\dagger. \quad (9)$$

The Kraus operators $\{\hat{M}_k\}$ can be deduced by the completely-positive master equation with memory kernel. By the method of [29], the i th column of the Kraus operator \hat{M}_k is the i th segment of the k th eigenvector of the completely-positive matrix \mathcal{P} . The Kraus operators are written as

$$\begin{aligned} \hat{M}_0 &= \sqrt{\mu_+}(a_+|e\rangle_S\langle e| + b_+|g\rangle_S\langle g|) \\ \hat{M}_1 &= \sqrt{1 - \xi(\gamma_0, t)}|g\rangle_S\langle e| \\ \hat{M}_2 &= \sqrt{\mu_-}(a_-|e\rangle_S\langle e| + b_-|g\rangle_S\langle g|), \end{aligned} \quad (10)$$

where the parameters $\mu_\pm = \{1 + \xi(\gamma_0, t) \pm \sqrt{[1 - \xi(\gamma_0, t)]^2 + 4\xi^2(\gamma_0/2, t)}\}/2$, $a_\pm/b_\pm = (\mu_\pm - 1)/\xi(\gamma_0/2, t)$ and $b_\pm = \xi(\gamma_0/2, t)/\sqrt{\xi^2(\gamma_0/2, t) + (\mu_\pm - 1)^2}$. For the Markovian reservoir

without memory, the Kraus operators can be simplified into $\hat{M}_0 = \nu|e\rangle_S\langle e| + |g\rangle_S\langle g|$ and $\hat{M}_1 = \sqrt{1-\nu^2}|g\rangle_S\langle e|$ where $\nu(t) = \exp(-\gamma_0 t/2)$.

3. Dynamics of quantum entanglement

As is known, decoherence of quantum entanglement actually happens when local measurements are applied in quantum information processing [2, 10]. A reasonable model is considered as one open qubit S entangled with an ancillary qubit A which is isolated from a local vacuum reservoir E [34]. The decoherence process can also be obtained by the map in terms of the total system-environment state [35, 36]. The evolution of the states for the total system can be obtained by

$$\rho(t) = U_{SE}(t)[\rho_{SA}(0) \otimes |0\rangle_E\langle 0|]U_{SE}^\dagger(t), \quad (11)$$

where $\rho_{SA}(0)$ is the initial entangled state. The unitary operator $U_{SE}(t)$ can be expressed by the Kraus operators,

$$U_{SE}(t)|m\rangle_S \otimes |0\rangle_E = \sum_k \hat{M}_k|m\rangle_S \otimes |k\rangle_E. \quad (12)$$

To clearly describe the dynamics of quantum entanglement between the system and local reservoir, we consider the negativity N [7] as one general measure for quantum entanglement. Based on the separability principle, the negativity N is introduced by,

$$N(\rho) = \max\{0, -2 \sum_i \epsilon_i\} \quad (13)$$

where ϵ_i is the i th negative eigenvalue of the partial transpose of the mixed state. For the separability of unentangled states, the partial transpose matrix has nonnegative eigenvalues if the corresponding state is unentangled.

When the maximally initial state $\rho_{SA}(0) = \frac{1}{2}(|ee\rangle + |gg\rangle)(\langle ee| + \langle gg|)$ is chosen, the reduced state $\rho_{SE}(t) = \text{Tr}_A[\rho(t)]$ at any time is written in the Hilbert space spanned by $\{|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle\}$,

$$\rho_{SA}(t) = \begin{pmatrix} \frac{1}{2}\xi(\gamma_0, t) & 0 & 0 & \frac{1}{2}\xi(\gamma_0/2, t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\xi(\gamma_0, t)}{2} & 0 \\ \frac{1}{2}\xi(\gamma_0/2, t) & 0 & 0 & \frac{1}{2} \end{pmatrix}. \quad (14)$$

The decaying of quantum entanglement N_{SA} is shown in Fig. 2(a). It is seen that the values of N_{SA} are always decreased to zero. The effects of memory kernel function on the entanglement are considered. When the relative amplitude α of the memory kernel $f(t)$ is decreased, the vanishing of the entanglement N_{SA} occurs more slowly. To clearly describe the decaying process, we can define the decoherence time τ where the value of the entanglement is declined to $1/e$. Fig. 2(b) clearly shows that the decoherence time scale is largely decreased with the increasing of the relative amplitude. We demonstrate that the decaying of quantum entanglement is much slower in the reservoir with the memory effects. This point is beneficial for the implementation of quantum computation in the non-Markovian environments.

It is also interesting to study the distribution of quantum entanglement between the systems and local reservoir. After the tracing of the ancillary system A , the reduced states between the open system S and the local environment ρ_{SE} can be expressed in the Hilbert space spanned by $\{|e0\rangle, |e1\rangle, |e2\rangle, |g0\rangle, |g1\rangle, |g2\rangle\}$,

$$\rho_{SE}(t) = \frac{1}{2} \begin{pmatrix} u & 0 & w & 0 & q_+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w & 0 & \xi(\gamma_0, t) - u & 0 & q_- & 0 \\ 0 & 0 & 0 & v & 0 & z \\ q_+ & 0 & q_- & 0 & 1 - \xi(\gamma_0, t) & 0 \\ 0 & 0 & 0 & z & 0 & 1 - v \end{pmatrix}, \quad (15)$$

where the elements are obtained by $u = \mu_+ a_+^2$, $v = \mu_+ b_+^2$, $w = \sqrt{\mu_+ \mu_-} a_+ a_-$, $z = \sqrt{\mu_+ \mu_-} b_+ b_-$ and $q_{\pm} = \sqrt{\mu_{\pm}} a_{\pm} \sqrt{1 - \xi(\gamma_0, t)}$. The negativity N_{SE} is also calculated and plotted as functions of the relative amplitude α and the time scale $\gamma_0 t$ in Fig. 3(a). It is shown that the sudden birth of entanglement occurs at some time in the decoherence process and then the values are rapidly decreased to zero after enough time. With the decreasing of α , the time interval for the increasing of N_{SE} is enlarged. Meanwhile, we also investigate

the dynamics of the reduced states ρ_{AE} after the tracing of the open system S ,

$$\rho_{AE}(t) = \frac{1}{2} \begin{pmatrix} u & 0 & w & 0 & 0 & 0 \\ 0 & 1 - \xi(\gamma_0, t) & 0 & p_+ & 0 & p_- \\ w & 0 & \xi(\gamma_0, t) - u & 0 & 0 & 0 \\ 0 & p_+ & 0 & v & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_- & 0 & 0 & z & 1 - v \end{pmatrix}, \quad (16)$$

where some elements are given by $p_{\pm} = \sqrt{\mu_{\pm}} a_{\pm}$. By the numerical calculation, the values of the entanglement N_{AE} are always increased to the maximum after enough time in Fig. 3(b). This means that the initial entanglement can be completely transferred to that between the ancillary system and local bath during the decoherence process. It is also found out that the entanglement between the ancillary system and independent bath is produced when the open system is coupled to the local reservoir.

4. Discussion

We investigate the decoherence model of the open two-level system coupled to a local vacuum reservoir using the Shabani-Lidar master equation with memory kernel. For the general exponential memory, the small amplitude of the kernel function and the large decaying rate are helpful to construct the completely-positive reduced map. To clearly describe the dynamics of quantum entanglement, we construct the efficient map in the Kraus representation. By means of the negativity, the entanglement distribution between the system and local environment is also studied. It is found out that the decreasing of quantum entanglement can be delayed pronouncedly in the non-Markovian reservoirs with memory effects. From the point of view of practical quantum information processing, the phenomenon is helpful for the understanding of the non-Markovian quantum computation. During the decoherence process, the sudden birth of entanglement between the open system and local bath happens at some time. After enough time, the initial entanglement will be transferred to that between the ancillary system and independent environment.

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Figure caption

Figure 1

(a). The difference D characterizing the complete positivity of the map is plotted as a function of α and $\gamma_0 t$ when the parameters are chosen by $\alpha \leq g = 0.5$; (b). The difference D is plotted when the parameters $\alpha \gg g = 0.1$.

Figure 2

(a). The dynamics of the entanglement between the open system and ancillary one N_{SA} is plotted when $g = 0.5$. The solid line denotes the case of $\alpha = 0.1$, the dashed one represents that of $\alpha = 0.3$ and the dotted line describes the case of Markovian reservoirs; (b). The decoherence time scale $\gamma_0 \tau$ is numerically calculated with the change of the relative amplitude α of the kernel function.

Figure 3

(a). The entanglement N_{SE} between the open system and local reservoir is plotted as functions of $\gamma_0 t$ and α when $g = 0.5$; (b) The entanglement N_{AE} between the ancillary system and reservoir is also plotted.

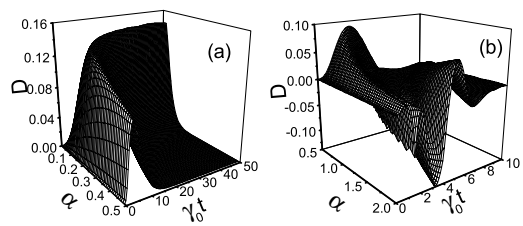


Fig. 1

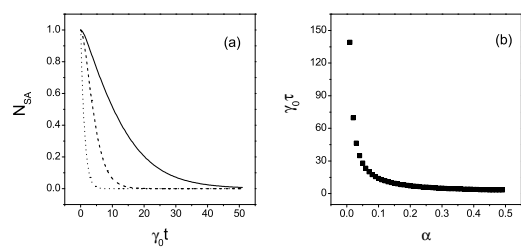


Fig. 2

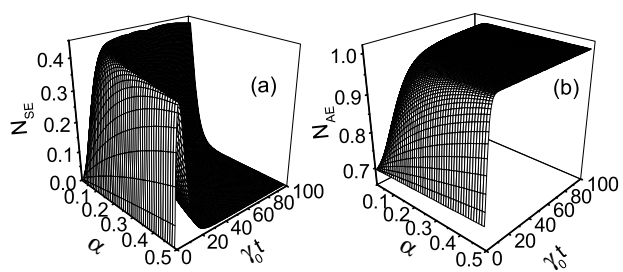


Fig. 3